

THE BERTINI INVOLUTION

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ABSTRACT. We summarize and extend E. Moody's results on the explicit equations related to the Bertini involution.

These notes are the result of my attempt to understand E. Moody's paper [1]. I correct a few misprints in [1] and take the computation a bit further.

I express my admiration to Ethel I. Moody, who managed to perform this tedious computation in the pre-Maple era. A Maple implementation of most equations is found at <http://www.fen.bilkent.edu.tr/~degt/papers/Bertini.zip>.

This text is not intended as an 'official' publication; it is distributed in the hope that it may be useful. It can be cited by its arXiv location.

Whenever possible, I try to keep the original notation of [1].

1. THE BERTINI INVOLUTION

1.1. **The results of [1].** Consider the pencil of cubics

$$(1.1) \quad \lambda w(x) + \mu w'(x) = 0,$$

where

$$w(x) = x_3^2(a_1x_1 + a_2x_2) + x_3(b_1x_1^2 + b_2x_1x_2 + b_3x_2^2) + (c_1x_1^2x_2 + c_2x_1x_2^2)$$

and similar for w' , so that the coordinate vertices are amongst the basepoints of the pencil. The point $(0 : 0 : 1)$ will play a special rôle.

The curve of the pencil passing through a point y is given by

$$(1.2) \quad W_3(x) := w(x)w'(y) - w'(x)w(y) = 0.$$

Clearly,

$$W_3(x) = x_3^2(A_1x_1 + A_2x_2) + x_3(B_1x_1^2 + B_2x_1x_2 + B_3x_2^2) + (C_1x_1^2x_2 + C_2x_1x_2^2),$$

where $A_i(y) := a_iw'(y) - a'_iw(y)$ and similar for B_i, C_i .

The tangent to (1.2) at $(0 : 0 : 1)$ meets the curve again at $r = (r_1 : r_2 : r_3)$, where

$$(1.3) \quad \begin{aligned} r_1 &= A_2r'_1, & r_2 &= -A_1r'_1, & r_3 &= A_1A_2r'_3, \\ r'_1 &:= B_1A_2^2 - B_2A_1A_2 + B_3A_1^2, & r'_3 &:= A_2C_1 - A_1C_2. \end{aligned}$$

The locus of these points is

$$(1.4) \quad \gamma_4(y) := y_1A_1 + y_2A_2 = 0.$$

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These notes are to be extended should there be any interesting development. They will be available at <http://www.fen.bilkent.edu.tr/~degt/papers/papers.htm> and on the arXiv.

Apart from the basepoints, the locus (1.4) meets (1.2) at a single point r . The line (ry) meets (1.2) at a third point z , and the *Bertini involution* can be defined as the map $y \mapsto z$. Let $\kappa := a_1 b'_1 - a'_1 b_1$ and

$$\begin{aligned} C_5(y) &:= A_2[B_1 + \kappa y_1 y_3^2]_{y_2} + [A_1 - \kappa y_1^2 y_3]_{y_2} [A_2 y_3 + B_3 y_2]_{y_1} + \kappa B_3 y_1 y_3, \\ \phi_6(y) &:= A_1 C_2 + y_3 C_5(y), \\ \psi_6(y) &:= A_2 C_1 + y_3 C_5(y). \end{aligned}$$

(Following [1], we use $[e]_u$ do indicate that e has a common factor u and this factor has been removed.) In these notations, the Bertini involution is

$$(1.5) \quad z_1 = \phi_6[A_2^2 \phi_6 + B_3 r'_1]_{y_1}, \quad z_2 = \psi_6[A_1^2 \psi_6 + B_1 r'_1]_{y_2}, \quad z_3 = \psi_6 \phi_6 C_5.$$

Apart from the basepoint $(0 : 0 : 1)$ of the pencil, the fixed point locus of this involution is the curve

$$(1.6) \quad K(y) := \psi_6[A_1 y_3 + B_1 y_1]_{y_2} - \phi_6[A_2 y_3 + B_3 y_2]_{y_1} = 0.$$

Remark 1.7. The expressions for r , C_5 , and K found in [1] contain a number of misprints. The corrections suggested are verified by the identities in §1.2 below, as well as by (2.2) and (2.3).

1.2. Further observations. The expression for the Bertini involution, see (3.1), is obtained by substituting $z = lr + my$ and solving $W_3(lr + my) = 0$, see (1.2), in $l : m$. (This equation is linear since $W_3(r) = W_3(y) = 0$.) Note that $\{\psi_6 = 0\}$ and $\{\phi_6 = 0\}$ are the curves contracted to the basepoints $(1 : 0 : 0)$ and $(0 : 1 : 0)$, respectively. Hence, they can also be found from the identities

$$y_3 r_1 - y_1 r_3 = A_2 \gamma_4 \phi_6, \quad y_2 r_3 - y_3 r_2 = A_1 \gamma_4 \psi_6.$$

Besides, one has

$$y_1 r_2 - y_2 r_1 = -r'_1 \gamma_4.$$

A point y is fixed by (3.1) if and only if the tangent at y to the member (1.1) of the pencil passing through y meets the curve again at r . In [1], the equation (1.6) of the fixed point locus is obtained by eliminating $\lambda : \mu$ from (1.1) and the polar conic to (1.1) with respect to r (after the substitution $x \mapsto y$). Alternatively, K can be found as the common factor of $y_3 z_1 - y_1 z_3$ and $y_2 z_3 - y_3 z_2$, using the identities

$$y_3 z_1 - y_1 z_3 = -\phi_6 K A_2, \quad y_2 z_3 - y_3 z_2 = -\psi_6 K A_1.$$

Note that the rightmost factors are just two particular members of the pencil.

2. THE MAP $\mathbb{P}^2 \dashrightarrow \Sigma_2$

From now on, we assume that the distinguished basepoint $(0 : 0 : 1)$ is simple.

2.1. The anti-bicanonical map. Let Y be the plane \mathbb{P}^2 blown up at all basepoints (including infinitely near) of the pencil other than $(0 : 0 : 1)$. It is a (nodal, in general) Del Pezzo surface of degree 1, and the anti-bicanonical linear system maps Y to a quadric cone in \mathbb{P}^3 . According to [1], the proper transforms of the sextics $\{\phi_6 = 0\}$ and $\{\psi_6 = 0\}$ are in $|-2K_Y|$. Hence, the space of sections $H^0(Y; -2K_Y)$ is generated by ϕ_6 (or ψ_6) and w^2 , ww' , w'^2 , and the map $y \mapsto \bar{z} \in \mathbb{P}^3$ is given by

$$\bar{z}_0 = \phi_6(y), \quad \bar{z}_1 = w^2(y), \quad \bar{z}_2 = w(y)w'(y), \quad \bar{z}_3 = w'^2(y).$$

Its image is the cone $\bar{z}_1 \bar{z}_3 = \bar{z}_2^2$. The passage to the affine coordinates $\bar{x} := \bar{z}_1 / \bar{z}_2$, $\bar{y} := \bar{z}_0 / \bar{z}_2$ blows up the vertex and maps the cone to the Hirzebruch surface Σ_2

with the exceptional section E of self-intersection (-2) (the exceptional divisor over the vertex). The composed rational map $\mathbb{P}^2 \dashrightarrow \Sigma_2$ is

$$(2.1) \quad \bar{x} = w(y)/w'(y), \quad \bar{y} = \phi_6(y)/w'^2(y).$$

Alternatively, Y with the remaining basepoint $(0 : 0 : 1)$ blown up is a rational Jacobian elliptic surface: the elliptic pencil is (1.1) and the distinguished section is the exceptional divisor over $(0 : 0 : 1)$. The Bertini involution becomes the fiberwise multiplication by (-1) , and the quotient blows down to the Hirzebruch surface Σ_2 . The quotient map is generically two-to-one; its ramification locus is the union of the exceptional section $E \subset \Sigma_2$ and a certain proper trigonal curve, *viz.* the image of $\{K = 0\}$. The pull-backs of the fibers of Σ_2 are the anti-canonical curves in Y (*i.q.* the members of the original pencil (1.1) of cubics), and the pull-backs of the *proper* (*i.e.*, disjoint from E) sections of Σ_2 are the anti-bicanonical curves other than those representable in the form $\{\alpha_1 w^2 + \alpha_2 w w' + \alpha_3 w'^2 = 0\}$.

2.2. The ramification locus. Since $\{\psi_6 = 0\}$ is the pull-back of a section of Σ_2 , there must be a relation (after the substitution $x \mapsto y$) of the form

$$(2.2) \quad \psi_6 = \phi_6 + S_2(w, w'),$$

where S_2 is a certain homogeneous polynomial of degree 2, see below.

The curve $\{\phi_6 = 0\}$ is contracted by the Bertini involution. Hence, the pull-back in Y of its image $\{\bar{y} = 0\}$ splits into two components (sections of the elliptic pencil), of which one is contracted by the blow down map $Y \rightarrow \mathbb{P}^2$. It follows that the free term R_3^2 in equation (2.3) below is indeed a perfect square.

Since $\{K = 0\}$ is the pull-back of the ramification locus (other than E), which is a proper trigonal curve, there must be a relation

$$(2.3) \quad K^2 = -4\phi_6^3 + \phi_6^2 P_2(w, w') + \phi_6 Q_4(w, w') + R_3^2(w, w'),$$

where P_2 , Q_4 , and R_3 are certain homogeneous polynomials of degree 2, 4, and 3, respectively. Let $S_2(t_1, t_2) = \sum_{i=0}^2 s_i t_1^i t_2^{2-i}$ *etc.* The coefficients p_i , q_i , r_i , s_i are found by a direct computation. They reduce to a remarkably simple form:

$$\begin{aligned} s_0 &= a_2 c_1 - a_1 c_2, \\ r_0 &= -a_1 b_2 c_2 + a_1 b_3 c_1 + a_2 b_1 c_2, \\ q_0 &= 4(a_1 c_2 - b_1 b_3) s_0 + 2b_2 r_0, \\ p_0 &= b_2^2 - 4a_2 c_1 - 4b_1 b_3 + 8a_1 c_2 \end{aligned}$$

and

$$p_i = (-1)^i \{p_0\}_i, \quad q_i = (-1)^i \{q_0\}_i, \quad r_i = (-1)^i \{r_0\}_i, \quad s_i = (-1)^i \{s_0\}_i,$$

where $\{\cdot\}_m$ is defined as follows: if e is a degree n monomial in a_1, \dots, c_2 , then $\{e\}_m$ is the sum of $\binom{n}{m}$ monomials, each obtained from e by replacing m of its n factors with their primed versions. (For example, one has $\{a_1 c_2\}_1 = a_1 c'_2 + a'_1 c_2$, $\{b_2^2\}_1 = 2b_2 b'_2$, and $\{a_1 b_1 c_1\}_2 = a_1 b'_1 c'_1 + a'_1 b_1 c'_1 + a'_1 b'_1 c_1$.) This definition extends to homogeneous polynomials by linearity.

Warning 2.4. The operation $\{\cdot\}_m$ is used only to shorten the notation. As with the derivative, this operation should be performed *before* any substitution of any particular values of the coefficients (see, *e.g.*, the substitution $a_1 = a_2 = 0$ in §3).

Remark 2.5. Observe that $S_2(w, w')$ remains unchanged under the transformation $a_i \leftrightarrow a'_i, b_i \leftrightarrow b'_i, c_i \leftrightarrow c'_i, w_i \leftrightarrow w'_i$. The same holds for $P_2(w, w')$ and $Q_4(w, w')$, whereas $R_3(w, w')$ changes sign.

Problem 2.6. The symmetry in Remark 2.5 is easily explained by interchanging w and w' . However, is there a geometric explanation for the ‘regular’ behaviour of the other coefficients?

Summarizing, we see that the map $\mathbb{P}^2 \dashrightarrow \Sigma_2$ given by (2.1) takes the sextics $\{\phi_6 = 0\}$ and $\{\psi_6 = 0\}$ to the sections $\{\bar{y} = 0\}$ and $\{\bar{y} = -S_2(\bar{x})\}$, respectively. The map is generically two-to-one (the deck translation being the Bertini involution), and its ramification locus in Σ_2 is the union of E and the trigonal curve

$$-4\bar{y}^3 + \bar{y}^2 P_2(\bar{x}) + \bar{y} Q_4(\bar{x}) + R_3^2(\bar{x}) = 0.$$

As usual, we treat the homogeneous bivariate polynomials S_2, P_2, Q_4 , and R_3 as univariate ones *via* $S_2(\bar{x}) := S_2(\bar{x}, 1) = \sum_{i=0}^2 s_i \bar{x}^i$ etc.

Problem 2.7. Can one express coefficients s_i in terms of p_i, q_i , and r_i ? In other words, does a choice of the ramification locus in Σ_2 and one of the sections select automatically the other section?

2.3. Other sextics contracted by the involution. The basepoint $(0 : 0 : 1)$ plays a special rôle in the definition of the Bertini involution. The other basepoints are not special. In particular, any other basepoint $(u_1 : u_2 : u_3)$ gives rise to a sextic $\{\psi_6^u = 0\}$ contracted to this point and to a splitting section of Σ_2 whose pull-back this sextic is. Assuming that $u_1 \neq 0$ and normalizing the coordinates as $(1, u_2, u_3)$, we have

$$(2.8) \quad \psi_6^u = \phi_6 + S_2^u(w, w'),$$

where S_2^u is a homogeneous polynomial of degree 2 whose coefficients s_i^u are

$$s_0^u = s_0 + (a_2 c_2 u_2 + (a_2 b_2 - a_1 b_3) u_3) + a_2 b_3 u_2 u_3 + a_2^2 u_3^2, \quad s_i^u = (-1)^i \{s_0^u\}_i.$$

As above, the image of $\{\psi_6^u = 0\}$ in Σ_2 is the section $\{\bar{y} = -S_2^u(\bar{x})\}$.

These equations are easily obtained by changing the coordinates and placing the basepoint in question to $(1 : 0 : 0)$.

3. THE GEISER INVOLUTION

Now, we assume that the pencil has exactly one basepoint infinitely near to the distinguished point $(0 : 0 : 1)$. In other words, all members of (1.1) have a common tangent at $(0 : 0 : 1)$ and, hence, exactly one of them has a double point at $(0 : 0 : 1)$. We can assume that this singular member is $\{w(x) = 0\}$, thus letting $a_1 = a_2 = 0$. The resulting special case of the Bertini involution is the *Geiser involution*, see [1].

3.1. The involution. Most formulas in §1.1 simplify dramatically. One obviously has $A_1 = -a'_1 w(y)$ and $A_2 = -a'_2 w(y)$; hence, $\gamma_4 = w\gamma_1$, see (1.4), where

$$(3.1) \quad \gamma_1 := -(a'_1 y_1 + a'_2 y_2)$$

is the defining polynomial of the common tangent to the members of the pencil at the distinguished basepoint $(0 : 0 : 1)$. (Here and below, w without an argument stands for $w(y)$.)

Next, there are splittings, see (1.3),

$$r'_1 = w^2 \tilde{r}'_1, \quad r'_3 = w \tilde{r}'_3, \quad r_i = w^3 \tilde{r}_i, \quad i = 1, 2, 3,$$

with

$$\begin{aligned}\tilde{r}_1 &= -a'_2\tilde{r}'_1, & \tilde{r}_2 &= a'_1\tilde{r}'_1, & \tilde{r}_3 &= a'_1a'_2\tilde{r}'_3, \\ \tilde{r}'_1 &= a'^2_2B_1 - a'_1a'_2B_2 + a'^2_1B_3, & \tilde{r}'_3 &= a'_1C_2 - a'_2C_1.\end{aligned}$$

Furthermore, one has

$$\phi_6 = w\phi_3, \quad \psi_6 = w\psi_3, \quad C_5 = w\tilde{C},$$

where

$$\begin{aligned}\tilde{C}(y) &:= -a'_2[B_1 - a'_1b_1y_1y_3^2]_{y_2} + a'_1[a'_2y_3(w - b_1y_1^2y_3) - B_3y_2]_{y_1y_2}, \\ \phi_3(y) &:= -a'_1C_2 + y_3\tilde{C}, \\ \psi_3(y) &:= -a'_2C_1 + y_3\tilde{C}.\end{aligned}$$

Finally, after reducing the common factor w^3 in (3.1), the Geiser involution takes the form

$$z_1 = \phi_3[a'^2_2w\phi_3 + B_3\tilde{r}'_1]_{y_1}, \quad z_2 = \psi_3[a'^2_1w\psi_3 + B_1\tilde{r}'_1]_{y_2}, \quad z_3 = \psi_3\phi_3\tilde{C}.$$

The loci contracted to the basepoints $(1 : 0 : 0)$ and $(0 : 1 : 0)$ are the cubics $\{\psi_6 = 0\}$ and $\{\phi_6 = 0\}$, respectively, and the fixed point locus is the sextic

$$\tilde{K}(y) := \psi_3[-a'_1wy_3 + B_1y_1]_{y_2} - \phi_3[-a'_2wy_3 + B_3y_2]_{y_1} = 0.$$

One has $K = w\tilde{K}$, see (1.6). The identities of §1.2 turn into

$$\begin{aligned}y_3\tilde{r}_1 - y_1\tilde{r}_3 &= a'_2\gamma_1\phi_3, & y_2\tilde{r}_3 - y_3\tilde{r}_2 &= a'_1\gamma_1\psi_3, & y_1\tilde{r}_2 - y_2\tilde{r}_1 &= -\tilde{r}'_1\gamma_1, \\ y_3z_1 - y_1z_3 &= a'_2\phi_3\tilde{K}, & y_2z_3 - y_3z_2 &= a'_1\psi_3\tilde{K}.\end{aligned}$$

3.2. The double covering $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Let Z be the plane \mathbb{P}^2 blown up at the basepoints (including infinitely near) of the pencil other than $(0 : 0 : 1)$ and the infinitely near one. It is a (nodal, in general) Del Pezzo surface of degree 2, and the anti-canonical linear system maps Z to \mathbb{P}^2 . This map is generically two-to-one, its deck translation is the Geiser involution, and its ramification locus is a quartic curve in \mathbb{P}^2 . The anti-canonical system is the web of cubics passing through the seven points blown up; the space $H^0(Z; -K_Z)$ is generated by any of ϕ_3 or ψ_3 and by w and w' . Hence, the corresponding rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $y \mapsto \bar{z}$, is given by

$$\bar{z}_0 = \phi_3(y), \quad \bar{z}_1 = w(y), \quad \bar{z}_2 = w'(y).$$

It is straightforward that $q_0 = r_0 = s_0 = 0$. Hence, there are splittings

$$Q_4(t, t') = t\tilde{Q}_3(t, t'), \quad R_3(t, t') = t\tilde{R}_2(t, t'), \quad S_2(t, t') = t\tilde{S}_1(t, t')$$

and relations (2.2) and (2.3) turn into

$$\begin{aligned}\psi_3 &= \phi_3 + \tilde{S}_1(w, w'), \\ \tilde{K}^2 &= -4\phi_3^3w + \phi_3^2P_2(w, w') + \phi_3\tilde{Q}_3(w, w') + \tilde{R}_2^2(w, w').\end{aligned}$$

Thus, the ramification locus is the quartic

$$(3.2) \quad 4\bar{z}_0^3\bar{z}_1 = \bar{z}_0^2P_2(\bar{z}_1, \bar{z}_2) + \bar{z}_0\tilde{Q}_3(\bar{z}_1, \bar{z}_2) + \tilde{R}_2^2(\bar{z}_1, \bar{z}_2).$$

In the coordinates chosen, the lines $\{\bar{z}_0 = 0\}$ and $\{\bar{z}_0 + \tilde{S}_1(\bar{z}_1, \bar{z}_2) = 0\}$ are double tangents (in the generalized sense) to this quartic.

Similarly, given another basepoint $(1 : u_2 : u_3)$, one has $S_2^u(t, t') = t\tilde{S}_1^u(t, t')$ and the cubic $\{\psi_3^u = 0\}$ singular at this point is given by (cf. (2.8))

$$\psi_3^u = \phi_3 + \tilde{S}_1^u(w, w').$$

Remark 3.3. The coefficients of the polynomials \tilde{S}_1 , \tilde{Q}_3 , and \tilde{R}_2 are the same as those of S_2 , Q_4 , and R_3 , respectively, see §2.2, with an obvious shift by one. It is worth emphasizing that the brace operation $\{\cdot\}_i$ should be evaluated *before* the substitution $a_1 = a_2 = 0$.

3.3. A few further observations. Unlike the general case considered in §2, now, the fixed point locus $\{\tilde{K} = 0\}$ does pass through $(0 : 0 : 1)$. In fact, since this curve is the branch set, it follows that $\{\tilde{K} = 0\}$ is also the locus of the singular points of the singular members of the web of cubics defined by the seven non-distinguished basepoints. This curve has a double point at each of the seven basepoints. In particular, it is an anti-bicanonical curve in Z .

Under the natural identification of the web and the dual plane $(\mathbb{P}^2)^\vee$, the locus of the singular members themselves is the curve dual to (3.2), including the lines through the singular points of (3.2).

As another observation, note that $\{\psi_3 = 0\}$ and $\{\phi_3 = 0\}$ are special members of the web, *viz.* those singular at $(1 : 0 : 0)$ and $(0 : 1 : 0)$, respectively. As above, these cubics are contracted by the involution to the corresponding basepoints.

REFERENCES

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